Signal Analysis and Communication ECE355

Ch. 1.3 : Exponential & Sinusoidal Signals

Lecture 3

13-09-2023

Ch. 1.3 EXPONENTIAL AND SINUSOIDAL SIGNALS

- they are fundamental building blocks for many general signals.
- they are mathematically easy to manipulate; differentiation & integration again yield complex exp. signals.

A. CT Complex Exponential & Sinusoidal Signals

- Defined as:

$$x(t) = Ce^{at} \; ; \quad C, a \in \mathbb{C} \qquad — \text{①}$$

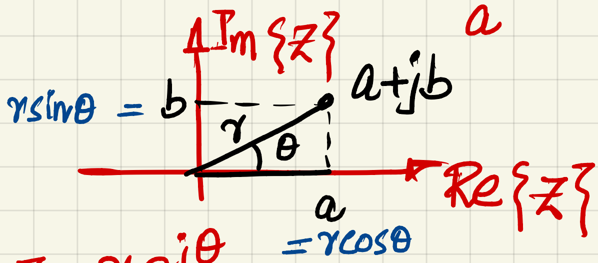
- Recall:

Complex numbers (Page 71)

$z = a + jb \; , \; j = \sqrt{-1}$

$r = |z| = \sqrt{a^2+b^2}$ amplitude/magnitude

$\theta = \measuredangle z = \arctan \frac{b}{a}$ angle/phase



$z = re^{j\theta}$ $\quad (z^* = re^{-j\theta})$

$= r(\cos\theta + j\sin\theta)$

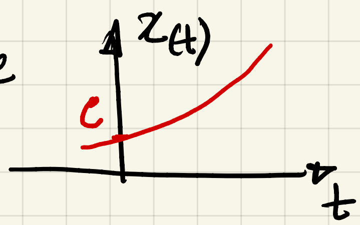
Also, $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$, $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{j2}$

3 Cases - ⓐ, ⓑ, and ⓒ

ⓐ Real Exponential

- If C & a are real in eqn ①

'a' +ve

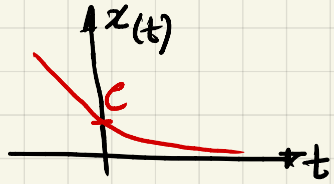


Examples

✓ Chain reaction in atomic explosion.

✓ Complex chemical reactions.

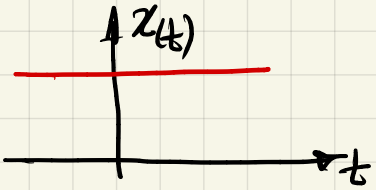
'a' -ve



Examples

- ✓ RC Circuit
- ✓ Radioactive decay.
- ✓ Damped mechanical circuit.

'a' 0



Example

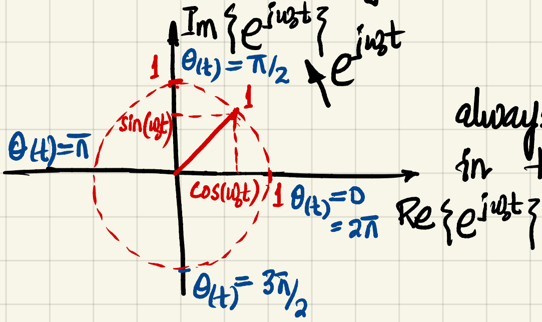
- ✓ DC Signal

(b) Periodic Complex Exponential & Sinusoidal

2 Cases - I & II

I. 'a' imaginary, 'c' real in eqn (1) ($a = j\omega_0$, $c = 1$ for simplicity)

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$



always lies on a unit circle in the complex plane.

$* \ \omega_0 t = \theta(t)$

- Important property - Periodic

$$\Rightarrow \quad e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$

- To be periodic

$$e^{j\omega_0 T} = 1$$

i. $\omega_0 = 0 \Rightarrow x(t) = 1$ Constant (periodic)

ii. $\omega_0 \neq 0$; In this case, to be periodic

$$\boxed{T_0 = \frac{2\pi}{|\omega_0|}} \quad \text{as} \quad e^{j2\pi} = 1$$

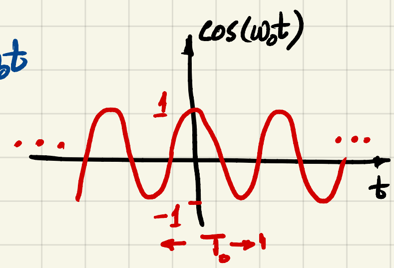
- Also, $e^{j\omega_0 T}$ & $e^{-j\omega_0 T}$ both have same period

Examples

$$x(t) = \cos(\omega_0 t)$$

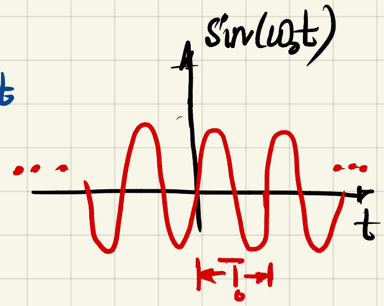
$$= Re\{e^{j\omega_0 t}\} = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$T_0 = 2\pi/|\omega_0|$$



$$x(t) = \sin(\omega_0 t)$$
$$= \text{Im}\{e^{j\omega_0 t}\} = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}$$

$T_0 = 2\pi/|\omega_0|$



II. 'a' imaginary, 'C' complex in eqn ①

($a = j\omega_0$, $C = |C|e^{j\phi}$)

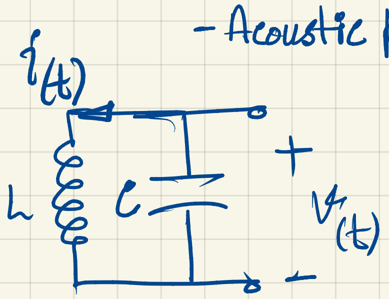
$$x(t) = |C|e^{j(\omega_0 t + \phi)}$$
$$= |C|e^{j(\omega_0 t + \phi)}$$

sinusoid advanced by ϕ/ω_0

$$= |C|\cos(\omega_0 t + \phi) + j|C|\sin(\omega_0 t + \phi)$$

- Periodic with period $T_0 = \frac{2\pi}{\omega_0}$

Examples: - physical sys. in which energy is conserved (LC circuit)
- mechanical sys.
- Acoustic pressure to single music tone.



$$\left.\begin{aligned} C\frac{dv}{dt} &= -i(t) \\ L\frac{di}{dt} &= v(t) \end{aligned}\right\}$$

$\omega_0 = \frac{1}{\sqrt{LC}}$

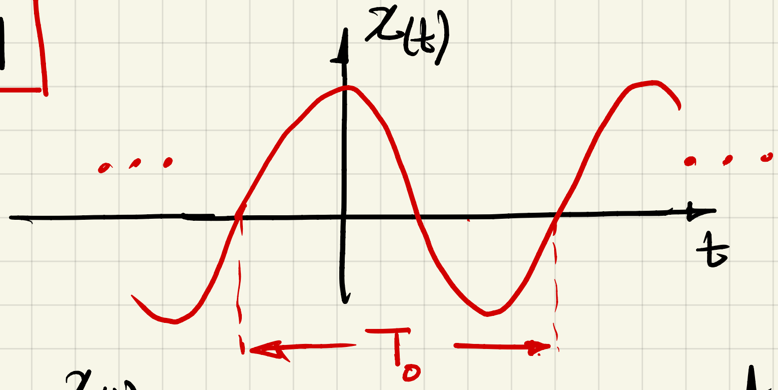
if $L = C$

$\omega_0 = \frac{1}{L} = \frac{1}{C}$

with initial conditions

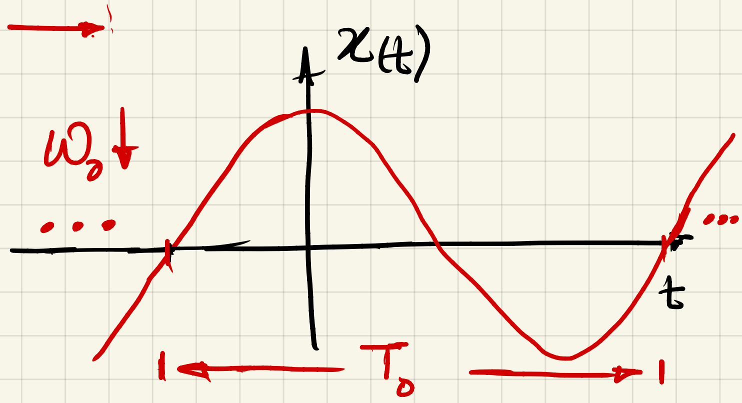
$v(0) = V_0$, $i(0) = 0$

$$\boxed{T_0 \propto \frac{1}{|\omega_0|}}$$



$\omega_0 \uparrow$

$\omega_0 \downarrow$



- For $\omega_0 = 0$ constant signal.

- Energy & Average Power of a Complex Exponential Sig.

$$E_{period} = \int_0^{T_0} |e^{j\omega_0 t}|^2 dt \qquad \text{(Case I)}$$

$$= \int_0^{T_0} 1 \times dt = T_0 \qquad \text{(finite)}$$

✗ $$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = \infty \qquad \text{(infinite)}$$

$$P_{period} = \frac{E_{period}}{T_0} = 1 \qquad \text{(finite)}$$

✓ $$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = 1 \qquad \text{(finite)}$$

© General Complex Exponential Sig.

- Both 'C' and 'a' are complex in eqn ①

$$C = |C| e^{j\theta} \qquad \text{polar form}$$

$$a = r + j\omega_0 \qquad \text{cartesian form}$$

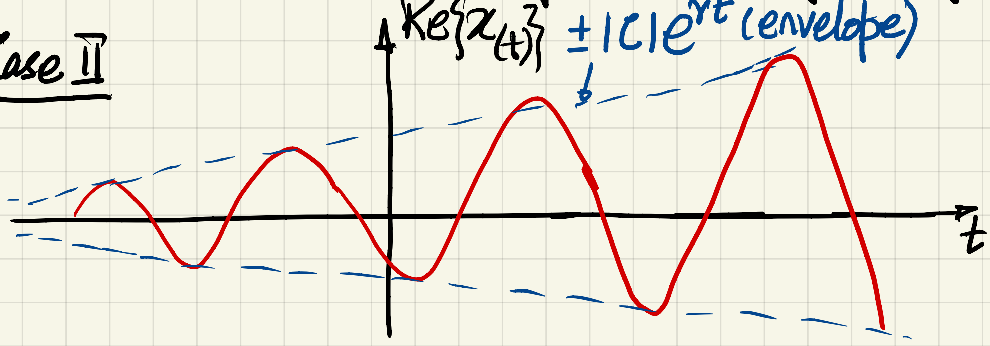
$$x_{(t)} = |C| e^{j\theta} e^{(r + j\omega_0)t}$$

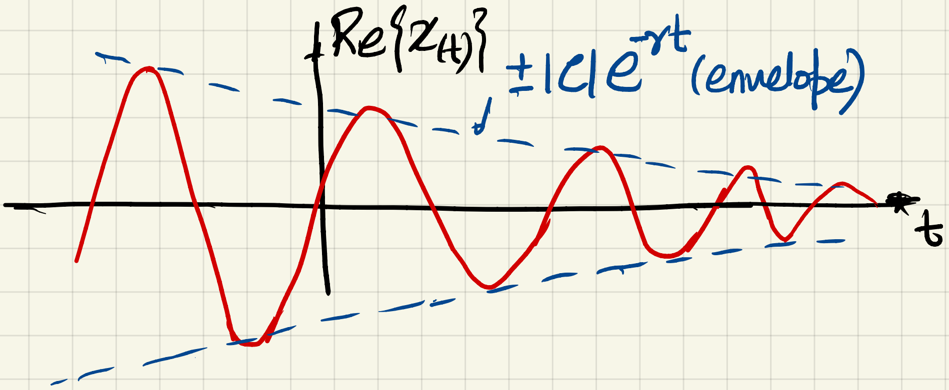
$$= |C| e^{rt} e^{j(\omega_0 t + \theta)}$$

Example: $Re\{x_{(t)}\} = |C| e^{rt} \cos(\omega_0 t + \theta)$

- Case I: $r = 0$ the real & imaginary parts - sinusoidal
- Case II: $r > 0$ Sinusoidal Sig. multiplied by a growing exponential
- Case III: $r < 0$ Sinusoidal Sig. multiplied by a decaying exponential

Case II





Examples: - Damped Sinusoid
- RLC Circuit (R dissipates energy)
- Mechanical Sys. (friction dissipates energy)
- Pushing a swing.

B. DT Complex Exponential & Sinusoidal Signals

- Defined as:

$$x[n] = C\alpha^n; \quad C, \alpha \in \mathbb{C} \quad - (2)$$ (convenient way to represent for DT)

OR

$$x[n] = Ce^{\beta n}; \quad C, \beta \in \mathbb{C} \quad - (3)$$ (analogous to CT defn)

where $\alpha = e^{\beta}$

3 Cases - (a), (b), and (c)

(a) Real Exponential Sig.

- if C & α are real in eqn (2)

Cases:

I. $|\alpha| > 1$ growing exp.
II. $|\alpha| < 1$ decaying exp.
III. $\alpha > 0$ $C\alpha^n$ - same sign for all 'n'
IV $\alpha < 0$ $C\alpha^n$ - sign alternates
V. $\alpha = 1$ C
VI. $\alpha = -1$ alternates b/w $+C$ & $-C$

Example: Total return on investment as a function of month/year.

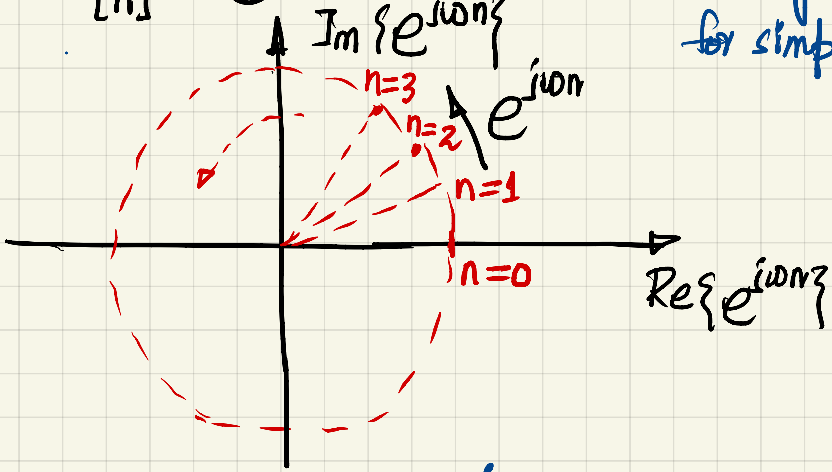
(b) Periodic Complex Exponential & Sinusoidal Sig.

2 Cases - I and II

I. 'β' imaginary and 'C' real in eqn (3)

$$x_{[n]} = e^{j\omega n}$$

(assuming $\beta = j\omega$ & $C = 1$ for simplicity)



Suppose $\omega = 1$, $x_{[n]} = e^{jn}$

- so we are hopping by one radian each time we increase n.
- however, if we hop by one radian, we will never again hop back to an integer multiple of 2π, because π is irrational.
- Therefore, this complex exp. is not periodic.

NOTE 1

* $x_{[n]} = e^{j\omega n}$ can be periodic or aperiodic depending on 'ω'!

So for what choices of 'ω' is $e^{j\omega n}$ periodic?

- Periodicity with period $N \in \mathbb{Z}_{\geq 1}$ requires that

$$x_{[n]} = e^{j\omega n} = e^{j\omega(n+N)} = e^{j\omega n} \cdot e^{j\omega N}$$

$\Rightarrow e^{j\omega N}$ should be '1'

$$e^{j\omega N} = 1 = e^{jk2\pi} \quad (k \in \mathbb{Z})$$

- it follows that we must have

$$\boxed{2\pi k = \omega N_0} \; *$$

* fundamental freq. is $\omega_0 = \frac{\omega}{k} = \frac{2\pi}{N_0}$

$\Rightarrow e^{j\omega n}$ is periodic with period 'N_0' if the frequency 'ω' is an integer multiple of $2\pi/N_0$ $\quad (k 2\pi/N_0)$

Examples: 1. $e^{j5\pi/6\, n}$ is periodic with period, $N=12$.

$\left(e^{j5\times\frac{2\pi}{12}n}\right)$

fundamental freq. is $\omega_0 = 2\pi/12 = \pi/6$

2. $x[n] = \cos\left(\frac{3\pi}{8}n\right)$

$= Re\left\{e^{j3\times\frac{2\pi}{16}n}\right\}$ is periodic with $N=16$.

NOTE 2

* Frequency-periodicity of DT Complex Exponentials $e^{j\omega n}$

- DT complex exponentials have another periodicity property.
- for any time 'n', note that

$$e^{j(\omega+2\pi)n} = e^{j\omega}\cdot\underbrace{e^{j2\pi n}}_{=1} = e^{j\omega n}$$

- $e^{j\omega n}$ is always a periodic function of 'ω' with period 2π.

$\Rightarrow$ DT complex exponentials are "periodic in frequency".

example: $e^{j\frac{5\pi}{6}n}$ and $e^{j\frac{17\pi}{6}n} = e^{j\left(\frac{5\pi}{6}+2\pi\right)n}$ are the exact same signal.

- Also, $$e^{j\omega n} = e^{j(\omega+2\pi k)n}, \quad k\in\mathbb{Z}$$

$\Rightarrow$ The sig. with freq. ω, $\omega\pm2\pi$, $\omega\pm4\pi$ are all identical.

- Therefore, we only consider freq. interval of length '2π' to choose 'ω'
  - Either $0\le\omega<2\pi$ ✓
  - Or $-\pi\le\omega<\pi$

- As $\omega\uparrow$ from zero, the rate of oscillation$\uparrow$ until we reach π.
- As $\omega\uparrow$ further, the rate of oscillation$\downarrow$ until we reach 2π.
- At $\omega=2\pi$, we have the same constant sequence as at $\omega=0$

$\Rightarrow$ Near $\omega=0, 2\pi$, even multiples of π – low freq. signals

& Near $\omega=\pm\pi$, odd multiples of π – high freq. signals.

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n \rightarrow \text{oscillates rapidly.}$$

II. 'β' imaginary, 'C' complex in eqn ③

$$\beta = j\omega, \quad C = |C|e^{j\phi}$$

$$\Rightarrow x_{[n]} = |C|e^{j\phi}e^{j\omega n}$$

$$= |C|e^{j(\omega n+\phi)}$$

$$= |C|\cos(\omega n+\underbrace{\phi}) + j|C|\sin(\omega n+\underbrace{\phi})$$

only phase shift
(time advance/delay (by ϕ/ω)) is the change here compared to Case I.

© General Complex Exponential Signal

- Both C & α are complex in eqn ②

$$C = |C|e^{j\theta}$$

$$\alpha = |\alpha|e^{j\omega}$$

polar

$$x_{[n]} = |C|\,|\alpha|^n e^{j\theta}e^{j\omega n}$$

$$= |C|\,|\alpha|^n e^{j(\omega n+\theta)}$$

$$= |C|\,|\alpha|^n[\cos(\omega n+\theta) + j\sin(\omega n+\theta)]$$

- Case I : $|\alpha| = 1$ Sinusoidal
- Case II : $|\alpha| > 1$ Sinusoidal sequence - growing exp.
- Case III : $|\alpha| < 1$ Sinusoidal sequence - decaying exp.